Computational Discovery with Newton Fractals, Bohemian Matrices, & Mandelbrot Polynomials

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There is also a transcript of an interview with these authors, conducted by Judy-anne Osborn.

See also Jürgen Gerhard **How to use Fibonacci numbers to teach recursive programming** Maple Transactions Volume 1, Issue 1, Article 14308 (July 2021). https://doi.org/10.5206/mt.v1i1.14308

See also the first Student Paper, Ewan Brinkman *et al.*, **The Theodorus Variation**, Maple Transactions Volume 1, Issue 2, Article 14500 (November 2021). https://doi.org/10.5206/mt.v1i2.14500

Context

- Teaching *Computational Mathematics* is increasingly important (Data Science, Visualization, Machine Learning, . . .)
- This is *difficult* because computational mathematics involves several things at once: mathematics (algebra, analysis, and more!), programming, complexity, and numerical stability, because of the *compromises* needed for efficiency.
- **Incorporating new things in the curriculum means removing old things from the curriculum** because we have only finite time to teach, and the students have only finite time to learn.
- **Active Learning** is by far the most demonstrably effective teaching method. See Scott Freeman *et al.*, "Active learning increases student performance in science, engineering, and mathematics". Proceedings of the National Academy of Sciences, 111(23):8410–8415, 2014.

We live in a rough world

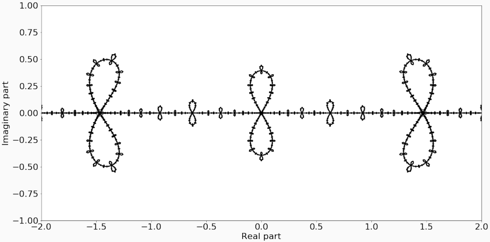


Figure 1: An infinite number of infinity symbols

"The existence of these patterns [fractals] challenges us to study forms that Euclid leaves aside as being formless, to investigate the morphology of the amorphous." (Benoit B. Mandelbrot, *The Fractal Geometry of Nature*, 1983, p. 1)

What we did/are doing

We have taught/are teaching courses in Computational Discovery (also known as Experimental Mathematics). At Western, the course mixed first-year students and graduate students (the grad students got one lecture per week extra, on advanced topics) in the *Western Active Learning Space*; this had pods, and fancy screens, and fancy connectivity; but the low-tech stuff worked very well, too.

So, what did we do, exactly? And why? Did it work?

Boy howdy, did it ever.

Sequences

Two of the most fundamental connections between programming and mathematics are the notions of *function* and of *iteration*. There is enough depth to these simple ideas that they cause many students great difficulty—a fact which those who grasp them easily *often overlook*. This is known as **Expert Blind Spot**.

We use the mathematical notion of *set* as a *data structure*.

Teaching these notions in a programming context helps to motivate (some) students. With first-year students, we can rely on their enthusiasm and idealism, and play games with sequences.

Mathematically, we are reinforcing the notion of *function*. Iteration is a function on the natural numbers!.

Guessing: What comes next?

Consider the sequence

$$1,\ \frac{3}{2},\ \frac{17}{12},\ \frac{577}{408},\ \ldots$$

What is the next term in the sequence? We feel that this particular sequence is actually too hard for the students to guess, so we don't ask them this—anyone here (who hasn't seen it before) want to give it a go?

Continued Fractions

What if we write the sequence as 1, $3/2 = 1 + [2]$, $17/12 = 1 + [2, 2, 2]$, $577/408 = 1 + [2, 2, 2, 2, 2, 2, 2]$, $\ldots$?

$$1 + [2, 2, 2] = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}$$

This is easier to guess. [We will come back to this example.]

Sneaking matrices in

Consider the 1 by 1 linear system $t_1 x_1 = 1$, the 2 by 2 linear system

$$\begin{aligned} t_1 x_1 + t_2 x_2 &= 1 \\ -x_1 + t_1 x_2 &= 0 \end{aligned} \tag{1}$$

and the 3 by 3 linear system

$$\begin{aligned} t_1 x_1 + t_2 x_2 + t_3 x_3 &= 1 \\ -x_1 + t_1 x_2 + t_2 x_3 &= 0 \\ -x_2 + t_1 x_3 &= 0\,. \end{aligned} \tag{2}$$

[The t_k are given, and we want to solve for the x_k.]

Teaching determinants

The sequence of determinants of those systems are t_1, $t_1^2 + t_2$, $t_1^3 + 2t_2 t_1 + t_3$,

$$t_1^4 + 3t_1^2 t_2 + 2t_1 t_3 + t_2^2 + t_4\ ,$$

and the 5 by 5 version has determinant

$$t_1^5 + 4t_1^3 t_2 + 3t_1^2 t_3 + 3t_1 t_2^2 + 2t_1 t_4 + 2t_2 t_3 + t_5\ .$$

The students have an interesting time trying to guess the pattern here, and even experts can be stumped: can you guess it?

That rational sequence again

We tell the students that the sequence 1, 3/2, 17/12, . . . is being generated by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) . \tag{3}$$

"Average your estimate with two divided by your estimate"; they can understand that if $x_n < \sqrt{2}$ then $2/x_n$ must be greater than $\sqrt{2}$ (and vice-versa), so the average of the two should be closer to $\sqrt{2}$.

Residuals

Consider the squares of that sequence: 1, 9/4, 289/144, $(577/408)^2 = \frac{332929}{166464}, \ldots$. Write them as $2-1$, $2+1/2^2$, $2+1/12^2$, $2+1/408^2$, …. Get the students to run the calculator to do the arithmetic, at least. The pattern becomes clear.

Now make the observation that these numbers are increasingly close to 2. That is, we have computed the exact square roots of numbers that are as close to 2 as we want. The students *get* this idea. This puts them in good position to experience the notion of *continuity*.

Consult the OEIS for the sequence of numerators: http://oeis.org/A001601. Denominators: http://oeis.org/A051009. There are an infinite number of places to go from here: let the students drive! This is very scary for both the students and the teachers.

If they don't want to drive just yet, then at least they are now ready to go on to Newton's method in general.

Pass the Parcel (Active Learning)

Newton's iteration in general is $z_{n+1} = z_n - F(z_n)/F'(z_n)$. To explain this to students we play the game of "pass the parcel": given an initial function (e.g. $F(z) = z^3 - 1$).

- One student chooses an initial number z_0 and passes it to the next. Here $n = 0$.
- The receiving student computes the next number by $z_{n+1} = z_n - F(z_n)/F'(z_n)$ and passes that result to the succeeding student.
- We go around the room until the iteration converges, or we get bored, or everyone has had a chance.

Students ask a crucial question

After this game has been played (maybe once is enough) you can pick a function and ask a student to start the game off.

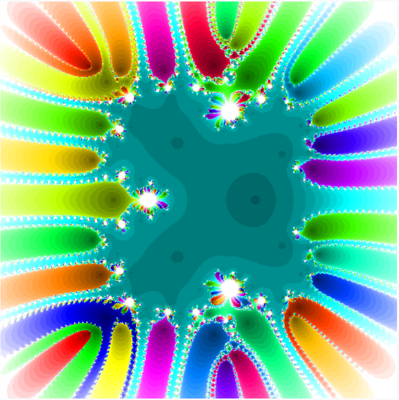
The student will ask "what initial estimate should I choose?"

You can throw that open to the class: it is an essential question. Once they have discussed it, you can show them what happens when you take every possible initial guess.

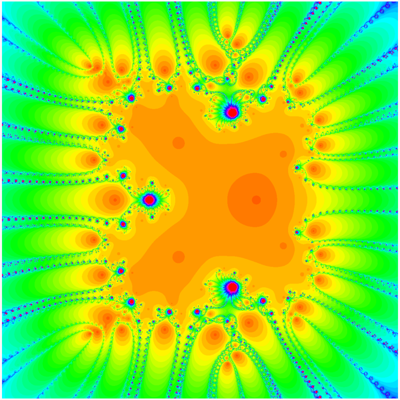
Maple has `Fractals:-EscapeTime:-Newton` for this, but it's relatively easy to write Python code for it as well.

Then let the students "go to town!"

A Newton fractal



(a) Python version



(b) Maple version

Figure 2: Newton fractals for a certain polynomial of degree 256.

Student-generated fractals

Figure 3: Forty student-generated fractals, using the Maple `Fractals:-EscapeTime:-Newton` package.

Challenges from IEEE floats

- "Admit, for instance, the existence of a minimum magnitude, and you will find that the minimum which you have introduced, small as it is, causes the greatest truths of mathematics to totter." — Aristotle
- Floats are *not associative*: $a + (b + c) \neq (a + b) + c$ necessarily. For instance $-M + (M + 1) = 0$ while $(-M + M) + 1 = 1$ if $M = 3.14 \cdot 10^{17}$.
- This (and other features) break students' models of how the world works.
- We have to enable students to deal with floats.

Mandelbrot polynomials

Define $z_0 = 0$ and $z_{n+1} = z_n^2 + c$. The set of c for which this iteration remains bounded is the famous *Mandelbrot set*.

If we do not define c as a complex number, then this sequence generates *polynomials in c*: $z_0 = 0$, $z_1 = c$, $z_2 = c^2 + c$, and so on. We can factor out one c and define $p_n(c) = z_n/c$ and then $p_{n+1} = cp_n^2 + 1$. The first few are $p_0 = 0$, $p_1 = 1$, $p_2 = c + 1$,

$$
\begin{aligned}
p_3 &= c^3 + 2c^2 + c + 1 \\
p_4 &= c^7 + 4c^6 + 6c^5 + 6c^4 + 5c^3 + 2c^2 + c + 1
\end{aligned}
$$

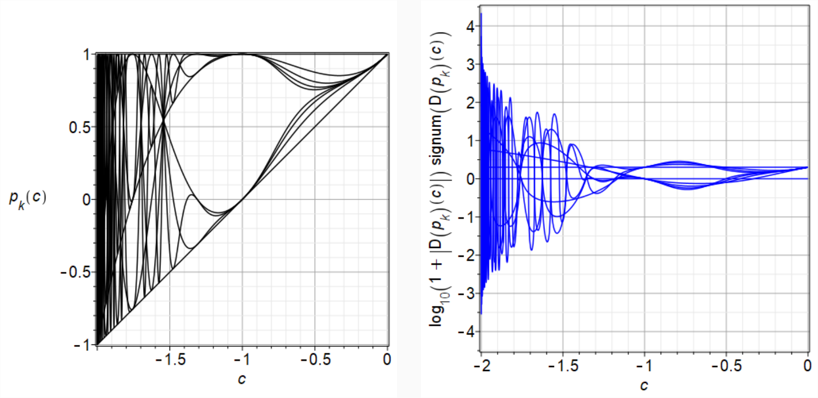
Zeros (roots) of these polynomials locate *periodic orbits* which are part of the Mandelbrot set.

Mandelbrot polynomials

```
function [p,dp]=mandelpoly(z,k)
% MANDELPOLY evaluates the k^th Mandelbrot polynomial
% and its derivative at one or more points.
% The k^th polynomial has degree 2^(k-1)-1

% Author PWL 2014.4.28 Modified RMC 2020.2.27
    dp = zeros(size(z));
    p  = zeros(size(z));
    for i=1:k-1
        dp = p.^2+2*z.*p.*dp;
        p  = z.*p.^2+1;
    end
```

Mandelbrot polynomials are numerically awkward



(a) $p_k(c)$, $k = 1, \ldots, 9$.

(b) $\mathrm{sign}(p_k'(c)) \log_{10} \left(1 + |p_k'(c)|\right)$

Figure 4: Derivatives of $p_k(c)$ are large: $p_k'(-2) = (4^{k-1} - 1)/3$.
http://oeis.org/A002450

Mandelbrot polynomial Newton fractal

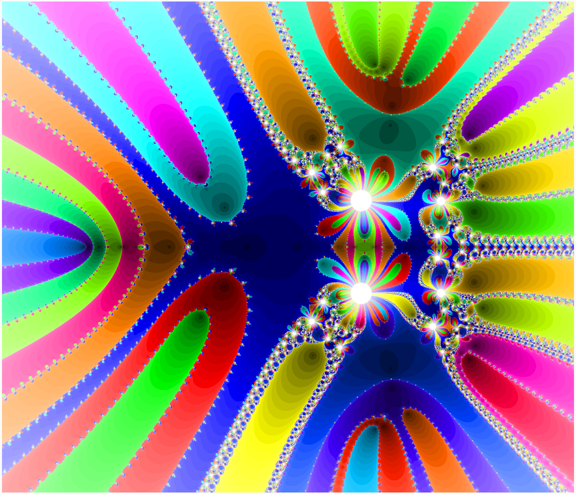


Figure 5: Newton fractal of $p_6(z)$ (generated using Python)

An open Mandelbrot polynomial conjecture

Are the coefficients of the Mandelbrot polynomials *unimodal*? Are they (apart from the n trailing coefficients, which are Catalan numbers) *log-concave*? We think so, but have no proof.

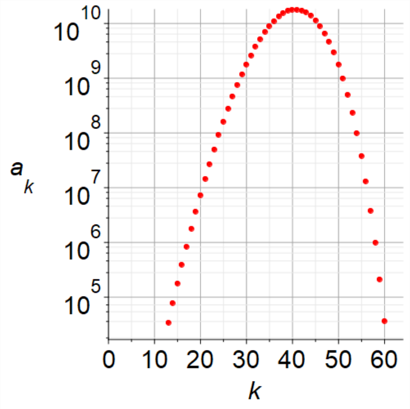


Figure 6: The coefficients of $p_7(c)$, the 7th Mandelbrot polynomial. It is degree 64.

Mathematical Notions Strengthened by Programming

Several mathematical notions are strengthened by these exercises.

- We use *mathematical induction* to prove correctness of the automatic differentiation of the Mandelbrot polynomials
- The analysis of IEEE floats uses the IEEE guarantees ($\mathrm{fl}(x \operatorname{op} y) = (x \operatorname{op} y)(1+\delta)$ for some $|\delta| \leq \mathbf{u}$ where $\mathbf{u}$ is the *unit roundoff*, 2^{-53} for double precision)
- Practice with functions is always useful
- Simply working with visualizations improves people's feel for geometry.

Matrices

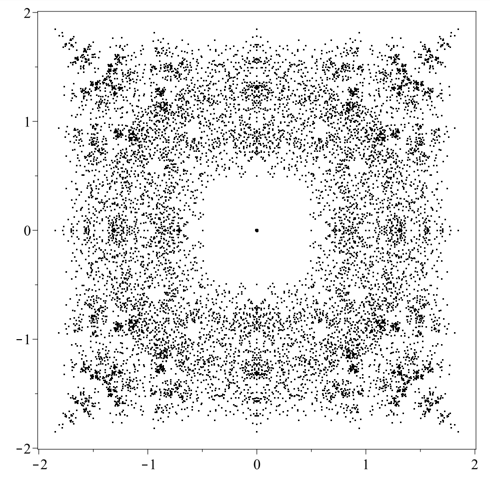


Figure 7: A Bohemian Example: All eigenvalues computed numerically by Maple of all 4096 seven by seven skew-symmetric tridiagonal matrices with entries from $\{1, i, 1+i, 1-i\}$. See bohemianmatrices.com

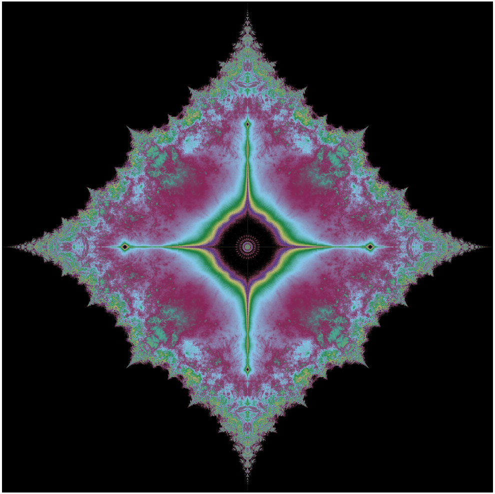


Figure 8: Another Bohemian Example: All eigenvalues of all 2^{30} (over a billion) 31 by 31 skew-symmetric tridiagonal matrices with entries from $\{1, i\}$. Picture by Aaron Asner (5 hours on a 32 core computer; C code)

Bohemian Matrices

To *avoid stealing the thunder of later courses*, we introduce determinants by the t_1–x_1 example stated earlier (anyone solve the puzzle yet?)

The topic of *Bohemian matrices* is particularly useful for this course: the name was invented in 2015 (the concept is *much* older) and it is only since then that so many open problems have been recognized. Some of the conjectures at http://www.bohemianmatrices.com/ have been resolved now, but by no means all; and it is *very* easy to generate more. Many of the pictures in that gallery were student-generated, by the way.

Here are two open conjectures:

1. How many m by m skew-symmetric tridiagonal matrices with population $[1, i, 1+i, 1-i]$ have multiple eigenvalues?
2. Does the asymptotic distribution of the eigenvalues converge to something as $m \to \infty$? Is it a uniform distribution on a disk? [Computation suggests ultimately uniform distribution on a square. But why a square??]

Why we teach it

The topic of *Eigenvalues* is incredibly important, and many introductory courses give it short shrift (and introduce it by characteristic polynomials anyway). The students simply need the practice.

We introduce the topic of *matrix structure*. They like this, and invent their own, e.g. "inverted checkerboard" matrices.

And younger students tend to be more creative (what does this say about our teaching?). Their creativity might actually be useful in this context, where everything is so new.

Breaking the mold

One of the pernicious myths that frustrates students and teachers alike is that “the answers are all in the book”. If that were true, why would we bother to teach mathematics?

So why not let students have some of the fun, and why not let them know straight away that there are questions they can attack that we don’t know the answers to?

These situations provide good opportunities to practice the “Action–Consequences–Reflection” principle [W. C. Bauldry, 2020].

Sunday's image

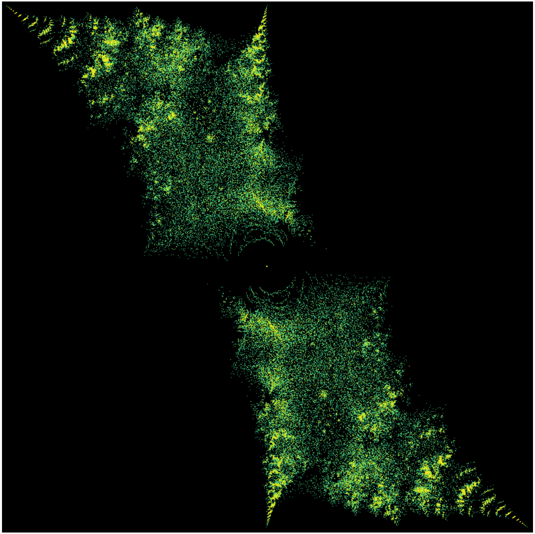


Figure 9: Density of eigenvalues of all 15 by 15 skew-symmetric tridiagonal Bohemians with population $[1, 1+i]$ (there are 16384 of them).

Assessment

How did we assess our students? We had no exams, only projects; interim reports were given to the class, and (for EYSC and RMC) student peer-grades were incorporated (averaged with the grades given by the instructor).

This course is not well-suited to a traditional exam.

Course learning outcomes included "programming ability," generally speaking. What we wanted the students to be able to do was to write and debug small programs with mathematical content: loops, possibly recursion, certainly conditionals; good programming practice like giving variables good names was important. Reproducibility mattered.

One memorable exercise was to mix up the teams during an in-class programming exercise, to teach them the value of documentation.

What we *really* wanted was the students to start getting comfortable asking their own questions.

Concluding Remarks

This course was designed with students' learning needs in mind, and also with student survey requests in mind ("more programming" and "more student control of the curriculum" are frequent responses).

Response from students was very enthusiastic.

NJC is still teaching his version of the course; he would say the same.

This course is, however, *hard to teach*, taking considerable design and execution effort; it is resource intensive. Whether it, or something like it, will fit in your program is up to you to decide.

Thank You For Listening.

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