The Lambert W function: Thirty Years Later

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Joint work with so very many people, but especially David Jeffrey
These slides available at rcorless.github.io

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Another announcement

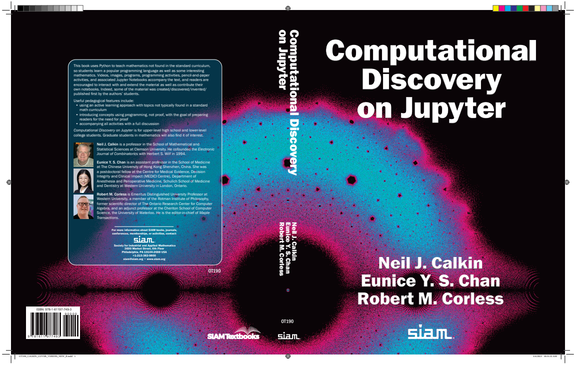

Figure 1: A 2023 book from SIAM: Calkin, Chan, & Corless, "Computational Discovery on Jupyter"

Yet another announcement

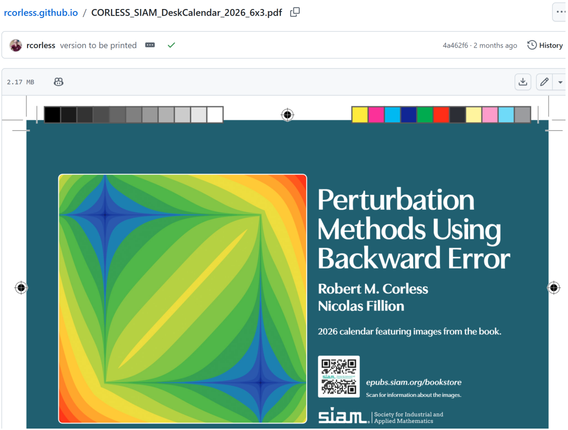

Figure 2: A 2026 book available April from SIAM: Corless & Fillion "Perturbation Methods Using Backward Error", and a calendar with images from the book

Como siempre

I have too many slides to “get through” in 50 minutes. We will do what we can, and stop on time! Ask questions whenever you like, and we may go off on tangents whenever we like.

The paper in question

The authors were myself, Gaston Gonnet, Dave Hare, David Jeffrey, and Donald E. Knuth.

A screenshot

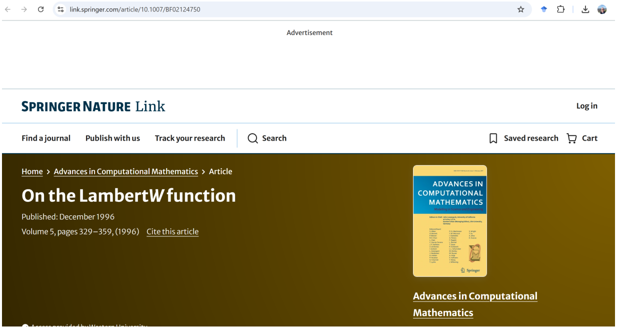

Figure 3: A screenshot from that link

Some "internal" historical details: TEX, not LATEX!

% First draft June 20, 1992 — RMC
% Second draft Sep 15+/92 — DEGH
% Third draft Oct 20+/92 —DEGH+RMC
% Final draft Dec 10+/92 — RMC+DJJ
% % Revisions (first cut after referee's report Aug '93)
% Feb 16+/94 — RMC+DJJ
% adding more 'glory' April 5, 1994 — RMC+DJJ
%
% taking away some of the glory, just to get the gosh-darned
% thing finished. November 4, 1994 — RMC+DJJ (T.J.Watson looking on)
%
% Final revision, after referee's comments, Feb 1996.
\font\titlefont=cmr17

Definition of W, T, W_k and T_k

If $y \exp y = x$, then y must be one branch of Lambert W. We say $y = W_k(x)$. The principal branch is $W_0(x)$, which we frequently write just as $W(x)$.

Donald Knuth prefers the "Tree T" function $T(x) = -W(-x)$ (for reasons which we will get to). There is no standard meaning for the branch numbering of the Tree T function, but I think it should be $T_k(x) = -W_{-k}(-x)$. [David Jeffrey agrees with me, or maybe I agree with him.] See the worksheet "behaviour.mw"

Calculus

$$\frac{dW}{dz} = \frac{1}{e^{W(z)}(1+W(z))} = \frac{W(z)}{z(1+W(z))} \tag{1}$$

unless $z = 0$ (take the limit, then, to get 1). Proof: $W \exp W = z$, so $W' \exp W + WW' \exp W = 1$ so if $z \neq 0$ then $W' = \exp(-W)/(1+W)$ $= W/(z(1+W))$. Our paper gives a recurrence relation for the general nth derivative using the exponential form, but Maple prefers the rational form.

$$\int W(x)\, dx = xW(x) + \frac{x}{W(x)} - x + C\,. \tag{2}$$

Proof: put $x = w \exp(w)$ so $dx = (1+w)\exp(w)dw$ and "Bob's your Uncle" because $\int w(1+w)\exp(w)\, dw$ can be done, for instance, by parts.

Power series

By the *Lagrange Inversion Formula*, or equivalently by using "suites of binomial type,"

$$W_0(x) = \sum_{n\geq 1}(-1)^{n-1}\frac{n^{n-1}}{n!}x^n \ . \tag{3}$$

This series converges if $|x| < 1/e$.

The Tree T function has the same series but without the pesky alternating signs:

$$T(x) = \sum_{n\geq 1}\frac{n^{n-1}}{n!}x^n \ . \tag{4}$$

This is the *generating function for the number of rooted trees with n nodes.*

Why W and not T, then?

It turns out that W itself appears (for instance) in the solution of the *delay differential equation*

$$\dot{y}(t) = zy(t-1)\,. \tag{5}$$

If $y(t) = H(t)$ on $-1 \le t < 0$ and $y(0) = y_0$ then this specifies $y(t)$ for all time. One can write

$$y(t) = \sum_n C_n e^{tW_n(z)} \tag{6}$$

where the C_n are determined by the initial conditions. The $W_n(z)$ that occur are *all* the branches of W evaluated at the parameter z.

But, really, both W and T are useful.

A distraction: series reversion by Cramer’s Rule

See the Jupyter notebook

Connections to the Gamma function

- The *functional inverse* of the Γ function can be approximated to an astonishing degree by using W: $\text{inv}\Gamma(x) = 1/2 + \ln p/W(\ln p/e)$ where $p = \ln(x/\sqrt{2\pi})$.
- The Puiseux series for W about the branch point $x = -1/e$ involve the *odd*-indexed coefficients in the formula classically known as Stirling's formula

It goes around the corner!

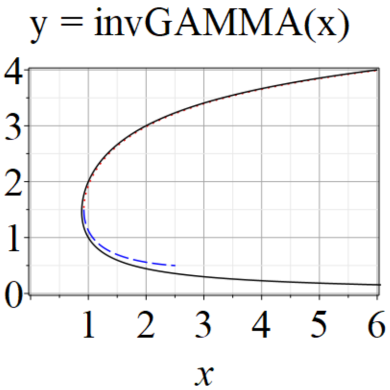

Figure 4: There are two real branches in that formula, giving astonishing accuracy

Connections to Spherical Bessel functions

A recent paper by David Stoutemyer

Branch Point Series

A recent paper by Gilbert Labelle

The slambangular integral

$$\frac{W(z)}{z} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1 - v\cot v)^2 + v^2}{z + v\csc v e^{-v\cot v}}\, dv \tag{7}$$

so long as z is not in the interval $(-\infty, -1/e]$.

Note: "slambangular" is a word invented by Bill Gosper when he saw this integral, and was astonished how fast the trapezoidal rule (or the midpoint rule) converged for it. He said it "converges slambangularly!" This integral is actually quite a good way to compute $W(z)$. It also shows that $W(z)/z$ is a *Stieltjes function*, which has many downstream effects: poles of the Padé approximants lie on the branch cut, for instance; and W of a matrix may be computed quickly.

The idea of the proof

The Cauchy integral formula says

$$\frac{W(z)}{z} = \frac{1}{2\pi i} \oint_C \frac{W(\zeta)}{\zeta(\zeta - z)}\, d\zeta \tag{8}$$

and we take a keyhole contour excluding $(-\infty, -1/e]$ but including $\zeta = z$. Let the radius of the contour go to infinity, and all that remains is the contributions from the top of the branch cut minus that on the bottom. Change variables $\zeta = w \exp w$ and the result follows.

That denominator

$$z + v \csc v e^{-v \cot v} = 0 \tag{9}$$

can be solved analytically in terms of branch differences of W. See Pseudospectra of exponential matrix polynomials for a proof.

$$v_{k,\ell} = \frac{1}{2i} \left(W_k(z) - W_\ell(z) \right) \,, \tag{10}$$

so long as the integer k is not the same as the integer ℓ. [As $v \to 0$, the function $v \csc v \exp(-v \cot v) \to 1/e$.]

A picture of those

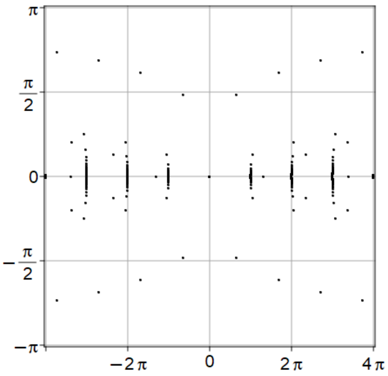

Figure 5: Those zeros have accumulation points at multiples of π. The zero at $v = 0$ is spurious; that is not a zero of the nonlinear equation. I should not have included $W_k(z) - W_k(z)$ in the list.

Accumulation points

Put $\ell = k \pm 1$ in the formula for $v_{k,\ell}$ and, for fixed z, let $k \to \infty$. Then $v \sim \pi \pm i/(2k) + O(\ln k/k^2)$. Letting $k \to -\infty$ gives the opposite sign. Using $k \pm 2$ gets $\pm 2\pi$, and so on.

Solving nonlinear equations can be arbitrarily difficult. This is one of the very few cases where we can explicitly see the difficulty. Numerical methods would fail pretty miserably.

Stirling's Original Formula

A paper in *Experimental Mathematics*

The arXiv version of that paper

Spider polynomials

See the November image in the calendar. These polynomials arise from the higher-order terms of the series for the functional inverse of the Γ function. The zeros of the polynomials give the picture in the calendar. See also Figure 5.4 in Corless & Fillion 2026. If $x = \Gamma(y)$ then

$$y = \frac{1}{2} + u_0 + \frac{1}{24u_0(1+W)} - \frac{\frac{1}{1152} + \frac{1}{576}(1+W) + \frac{7}{2880}(1+W)^2}{u_0^3(1+W)^3} + \cdots \quad (11)$$

where $u_0 = \ln p / W(e^{-1} \ln p)$ and $p = x/\sqrt{2\pi}$.

The first few

We do not know if these polynomials occur in other contexts.

$$
\begin{aligned}
&1\\
&5+10\,(1+W)+14\,(1+W)^2\\
&105+350\,(1+W)+714\,(1+W)^2+1176\,(1+W)^3+2232\,(1+W)^4\\
&2625+12250\,(1+W)+32970\,(1+W)^2+68040\,(1+W)^3\\
&\qquad+136956\,(1+W)^4+292536\,(1+W)^5+822960\,(1+W)^6 \qquad (12)
\end{aligned}
$$

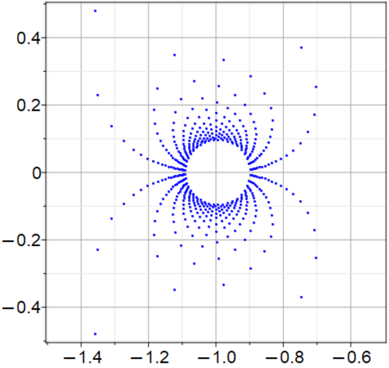

Figure 6: The roots of the first 22 spider polynomials

Condorcet's problem

The Wikipedia article on the Marquis de Condorcet (1743–1794)

If

$$y^{y^{y^{\cdot^{\cdot^{\cdot}}}}} = z \tag{13}$$

can we find a "closed form" for y? Yes.

$$y = -\frac{W(-\ln z)}{\ln z} \,. \tag{14}$$

Moreover, the iteration converges if $|W(-\ln z)| < 1$ (this is *Carlsson*'s theorem from 1907).

Carlsson's Theorem and Baker & Rippon

Baker & Rippon 1985 give a hand-drawn fractal. My version by computer which I think illustrates Carlsson's 1907 theorem better. The plot is of ζ where $\zeta \exp(-\zeta) = \ln z$, or $\zeta = T(\ln z) = -W(-\ln z)$; in fact we use the 0 branch. The colours indicate the period of the cycle the iteration $a_{n+1} = z^{a_n}$ with $a_0 = 0$ tends to. Inside $|\zeta| < 1$, the iteration converges to a 1-cycle; that is, a definite limit. This is Carlsson's Theorem. Baker & Rippon proved that it also converged at *rational* angle points on $|\zeta| = 1$; that is, $\zeta = \exp(2\pi i s)$ for rational s; and *only* for rational s.

A better version (2023)

A closer look

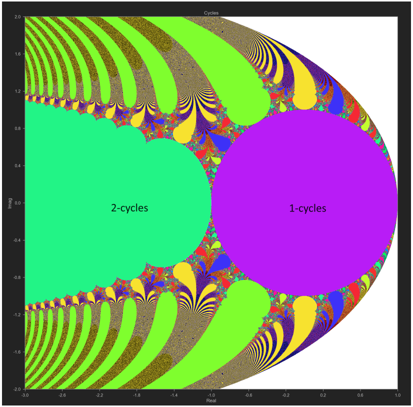

Figure 7: The fractal tower

Those scalloped edges

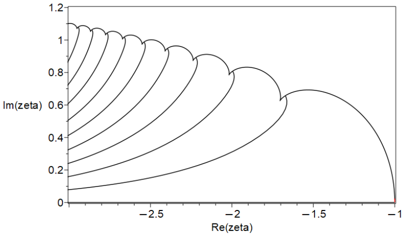

Figure 8: The regions where different branches of *W* play a role; note the corners are not where the branches change! The corners are due to *lack of conformality* of the map.

Applications

Some mostly random citations from Google Scholar

- Hofmann, K.P. and Lamb, T.D., 2023. Rhodopsin, light-sensor of vision. Progress in Retinal and Eye Research, 93, p.101116.
- Low, G.H. and Chuang, I.L., 2017. Optimal Hamiltonian simulation by quantum signal processing. Physical review letters, 118(1), p.010501.
- Wang, C., 2023. Calibration in deep learning: A survey of the state-of-the-art. arXiv preprint arXiv:2308.01222.
- Higham, N.J., 2008. Functions of matrices: theory and computation. Society for Industrial and Applied Mathematics.

$$\omega(z) = W_{K(z)}\left(e^z\right) \tag{15}$$

where $K(z)$ is the *unwinding number* with $\ln e^z = z - 2\pi i K(z)$. The function $\omega(z)$ is single-valued, and unless $z = t \pm i\pi$ with $t \leq -1$, solves $y + \ln y = z$.

$$W_k(z) = \omega(\ln_k z) \tag{16}$$

where $\ln_k z = \ln z + 2\pi i k$ is David Jeffrey's notation. The logarithm when $k = 0$ is the principal branch with $-\pi < \arg(z) \leq \pi$.

So much more to say

There are series at infinity, one can use W on matrices, and on and on and on.

But I will stop here.

Thank you!

Happy to take questions!

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