The maximum spread of symmetric Bohemian matrices

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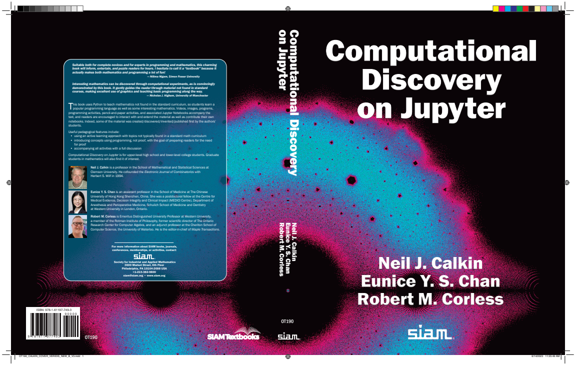

Figure 1: from SIAM: Calkin, Chan, & Corless, “Computational Discovery on Jupyter”, published November 2023

Has some background on Bohemian Matrices.

Bohemian Matrices

A family of matrices is called "Bohemian" if all entries are all from a single finite population P. The name comes from BOunded HEight Matrix of Integers. See bohemianmatrices.com for instances.

See also the [link] London Mathematical Society Newsletter, November 2020, page 16.

Such matrices have been studied for quite a long time (e.g. by Olga Taussky–Todd), though the name "Bohemian" only dates to 2015. See also the Wikipedia entry at `https://en.wikipedia.org/wiki/Bohemian_matrices`.

A partial list of some related work

- Chan, E. Y.S., Corless, R. M., Gonzalez-Vega, L., Sendra, J. R., Sendra, J., & Thornton, S. E. (2020). Upper Hessenberg and Toeplitz Bohemians. LAA, 601, 72-100.
- Chan, Eunice YS, Robert M. Corless, Laureano González Vega, and Juana Sendra. "Bohemian matrices: A source of challenges." In EACA 2022: XVII Encuentro de Álgebra Computacional y Aplicaciones, pp. 59-62. Servei de Comunicació i Publicacions, 2023.
- Robert Corless, George Labahn, Dan Piponi, and Leili Rafiee Sevyeri. 2022. Bohemian Matrix Geometry. ISSAC '22 ACM 361–370.

More

- Calkin, Neil J., Eunice YS Chan, Robert M. Corless, David J. Jeffrey, and Piers W. Lawrence*. "A fractal eigenvector." The American Mathematical Monthly 129, no. 6 (2022): 503-523.

*Piers W. Lawrence (April 13, 1987–July 2, 2025) contributed one of the foundational ideas of the Bohemian matrix project, namely the minimal height Mandelbrot matrix.

Definitions

Let A be a square matrix of dimension m with entries in $\mathbb{R}$. The spread of A is defined as the maximum of the distances between the eigenvalues of A.

$$\mathrm{spread}(A) := \max_{i,j} |\lambda_i - \lambda_j| \tag{1}$$

This quantity occurs in several places (for instance in theorems about convergence of some iterative algorithms).

Preliminaries (from Rafa's talk at ILAS 2023)

- Computationally: one can approximate the spread
- Theoretically: upper and lower bounds of $\mathrm{spread}(A)$.
    - Mirsky Bound (1956): $n \geq 3$

$$\mathrm{spread}(A) \leq \sqrt{2\|A\|_F^2 - \frac{2}{n}|\mathrm{Tr}(A)|^2}.$$

Moreover, the equality holds iff A is normal and $n-2$ of its eigenvalues are equal and this common value is the arithmetic mean of the two others.

Real symmetric matrices

We are concerned here with real symmetric matrices, with entries from a fixed interval $[a,b]$. All eigenvalues are then real, and can be sorted to $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_m$. The spread is then $\mathrm{spread}(A) = \lambda_1 - \lambda_m$.

Nondimensionalizing

Put $\mathsf{S}_1 = \mathsf{A}$ if $|a| \le |b|$. If $|a| > |b|$ then put $\mathsf{S}_1 = -\mathsf{A}$ instead, with elements from the interval $[-b, -a]$. The eigenvalues are negated but the spread remains the same. Now consider $\mathsf{S} = \mathsf{S}_1/\max(|a|,|b|)$. The entries of S are in the interval $[\alpha,1]$ where $-1 \le \alpha$ because every entry is no bigger in magnitude than 1. The spread of S is $\mathrm{spread}(\mathsf{S}_1/\max(|a|,|b|))$.

Therefore we may reduce consideration to intervals $[\alpha,1]$ with $-1 \le \alpha$.

Reduction to Bohemian matrices

Biborski reduced the the problem by showing that the maximal spread would occur when the matrix had entries belonging to the set $\{a,b\}$. That is, only the extremal points of the interval $[a,b]$ were needed. So we only need to consider symmetric matrices with entries α or 1.

- Biborski, Iwo, Note on the spread of real symmetric matrices with entries in fixed interval, LAA 632 246–257 (2022)

A conjecture

In 2012, Fallat & Xing **conjectured** that the maximum spread of a real symmetric matrix of dimension m would occur for a *rank two matrix*. This made explicit computation of that putative maximum possible: all rank-two matrices in this family are permutationally similar to a matrix of the form

$$\begin{bmatrix} \alpha_{k\times k} & \mathsf{1}_{k\times(m-k)} \\ \mathsf{1}_{(m-k)\times k} & \mathsf{1}_{(m-k)\times(m-k)} \end{bmatrix}$$

with some block size k with $1 \leq k \leq m-1$.

The conditions for the Mirsky bound *nearly* hold: one positive eigenvalue, $m-2$ zero eigenvalues, and one negative eigenvalue: but 0 is *not* the average of the $+$ and $-$ eigenvalues.

The explicit formula

It's a fun exercise to show that the eigenvectors of that rank two matrix necessarily have a fixed form

$$\mathrm{spread}(\mathsf{S}) = \sqrt{(\alpha^2 + 2\alpha - 3)k^2 + 2m(1-\alpha)k + m^2}\,.$$

Moreover, the block size k was explicitly given by Fallat & Xing:

$$k = \mathrm{round}\left[\frac{m}{\alpha + 3}\right]\,.$$

The current state of knowledge

The conjecture is not known to be generally true. Zhan had already proved that it's true if $\alpha = -1$ in 2005. If $\alpha = 0$ then Breen et al proved in 2022 a result that implies that the conjecture is true if $m \cong 0 \mod 3$.

But we do not know if the conjecture is true in general.

Our results

- If $\alpha = 0$ then the conjecture is true for dimensions $m = 2, 3, 4, 5, 6, 7$, and 8.
- For α in $(-1,1)$ the conjecture is true for dimensions $m = 2, 3, 4, 5, 6$, and 7.

Proof: Exhaustive exact and symbolic computation.

Some details

The number of symmetric matrices with 2 different possible entries is $2^{m(m+1)/2}$. This grows (much) faster than factorial! Up to $m = 8$ the numbers are 2, 8, 64, 1.024, 32.768, 2.097.152, 268.435.456, 68.719.476.736. Computing exactly the eigenvalues of more than 68 billion eight by eight matrices is not a smart way to try to solve this problem.

We might think about *resultants* of the characteristic polynomials: The resultant $R(T) = \mathrm{res}_\lambda(p(\lambda), p(\lambda + T))$ whose roots are the differences $\lambda_i - \lambda_j$. Then we might look for the resultant with the largest magnitude root.

But $m = 4$ is instructive

There are only 1.024 different four by four matrices with entries either α or 1. When $\alpha = 0$ this results in 52 unique nontrivial resultants. Many, many matrices in this class have the *same* characteristic polynomial. Once we have the resultants, we can use exact rational interval arithmetic to get guaranteed bounds on the spread and prove that no matrix has greater spread than the rank-two matrix of the Fallat & Xing conjecture. This is brutal, but it works.

The symbolic case when $m = 4$

If α is a symbol, then the roots T of the resultant are functions of α. This means more symbolic work. When $m = 4$ there are only 77 distinct nontrivial resultants. If we plot the difference between the roots of these and the formula for the conjectured maximum we get the graph in the next frame.

An unconvincing visual

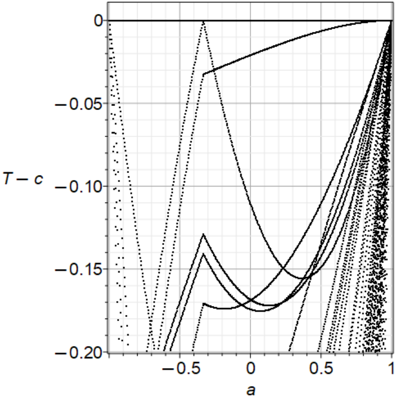

Figure 2: The difference between the computed spread of each of the 77 classes of matrices of dimension $m = 4$ and the conjectured maximal spread, at 501 equally-spaced discrete values of a in the interval $-1 < a < 1$. No counterexamples to the conjecture are detected, although the graph is ambiguous near $a = 1$ where one of the curves appears to approach the x axis.

An A-Eye is needed

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \alpha \end{bmatrix}. \tag{2}$$

The spread of this matrix is $(1-\alpha)/2 + \sqrt{\alpha^2 - 2\alpha + 17}$. Comparing this to the conjectured maximal spread we see that it is smaller, intersecting only at $\alpha = 1$. However, it gets very close indeed: the slope of this curve at $\alpha = 1$ is exactly the same as the slope of the conjectured maximal spread curve, at $\alpha = 1$: both are $-1/2$. We have therefore demonstrated by example that matrices of rank other than 2 can have spread arbitrarily close to the maximal spread.

Our A-Eye uses Sturm Sequences

We wrote an “Artificial Eye” that examines each resultant and decides if, anywhere on the interval, its largest root is larger than the conjectured maximum. The code is not long, but relies on several well-tested pieces of Maple.

The question remains of what to do about the massive growth in number of matrices.

Graph theory to the rescue

In the case of symmetric matrices whose elements are either 1 or a where a is not necessarily 0, we may write

$$S = E + (a - 1)G \tag{3}$$

where $\boldsymbol{E} = \boldsymbol{e}\boldsymbol{e}^T$ is the matrix of all 1s, $\boldsymbol{e}$ is the column vector of all 1s, and $\boldsymbol{G}$ is a symmetric 0–1 matrix, which we may interpret as an adjacency matrix for a graph, again with the possibility of self-loops.

Every graph isomorphism can be represented as a permutation similarity of $\boldsymbol{G}$.

Practical graph isomorphism

Brendan D. McKay and Adolfo Piperno, Practical graph isomorphism, II, JSC, 60, 94-112, 2014 describes the *nauty* package, some parts of which are hooked into the Maple GraphTheory package. We really wanted the digraphs part, but we could make the graphs part work.

Reducing the number

```
seq([j, (2**j - 1)*GraphTheory:-NonIsomorphicGraphs(j, output = count)], j = 1 .. 9);
```

```
[1, 1], [2, 6], [3, 28], [4, 165], [5, 1054], [6, 9828], [7, 132588], [8, 3148230], [9, 140355348]
```

This is for $\alpha = 0$. Still, when $m = 8$ we have only just over 3 million matrices, not 68 billion. So it's a massive improvement.

The rest is brutality

When $\alpha = 0$, the computations for $m = 8$ took 1.2 hours on a Microsoft Surface Pro. The computations for $m = 9$ would take a week at least, and would not tell us anything new (there is a proof for $m = 3\ell$ and $\alpha = 0$). For symbolic α, the computations for $m = 7$ took 36 hours on the same machine. The case $m = 8$ would take more than a month, if they succeeded at all. I'm looking to use a bigger machine for this case. The code is available on request.

Thank you

Thank you for listening!

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