Teaching Mathematics in a Mechanized Environment

Robert M. Corless

The Ontario Research Centre for Computer Algebra, The Rotman Institute of Philosophy, and The School of Mathematical and Statistical Sciences, Western University, Canada
Cheriton School of Computer Science, University of Waterloo, Canada
Editor-in-Chief, Maple Transactions

Announcement: Maple Transactions

Maple Transactions

an open access journal with no page charges

mapletransactions.org

We welcome expositions on topics of interest to the Maple community, including in computer-assisted research in mathematics, education, and applications. Student papers especially welcome.

Context

- Teaching *Computational Mathematics* is increasingly important (Data Science, Visualization, Machine Learning, . . .)
- This is *difficult* because computational mathematics involves several things at once: mathematics, programming, complexity, and numerical stability, because of the *compromises* needed for efficiency.
- **Incorporating new things means removing old things** because we have only finite time to teach, and the students are learning other things as well.
- **Active Learning** is by now recognized as being by far the most effective way to learn.

Isn’t technology taken into account already?

No, at least not everywhere.

There have been arguments for at least fifty years about this. Maybe the best paper on the subject is Bruno Buchberger’s 1990 paper “Should Students Learn Integration Rules?” [Link] where he summarizes a resolution of previous fierce discussions with colleagues into what is known now as the “White Box/Black Box Principle”. [I will give details shortly.]

But not everyone knows this. Let’s look first at some **recent bad examples** (from my own University)

A question from a 1st-year calculus exam, Western 2022

The exam is online at studocu.com; this is the *first* question I looked at.

"Which of the following statements most accurately describes the improper integral $\int_{x=1}^{\infty} \frac{1}{x(2x+1)}\, dx$:

a) divergent
b) convergent to $\ln(1/3)$
c) convergent to $\ln(2/3)$
d) convergent to $\ln(3)$
e) convergent to $3\ln(3)$"

Narrator: **all** the given choices are wrong. Both Maple and SymPy give the correct answer of $\ln(3/2)$. **Every instructor in the audience will know what a headache this kind of blunder can cause.**

It's actually a bad question for another reason too

Partial fractions gives

$$\frac{1}{(2x+1)x} = \frac{1}{x} - \frac{1}{x+\frac{1}{2}}$$

which splits the convergent integral into two divergent ones. Some students won't notice and will blithely compute

$$\begin{aligned}
\int_{x=1}^{\infty} \frac{dx}{(2x+1)x} &= \int_{x=1}^{\infty} \frac{dx}{x} - \int_{x=1}^{\infty} \frac{dx}{x+\frac{1}{2}} \\
&= \ln x\Big|_{x=1}^{\infty} - \ln(x+\frac{1}{2})\Big|_{x=1}^{\infty} \\
&= \infty - 0 - \infty + \ln\frac{3}{2} \\
&= \ln\frac{3}{2}
\end{aligned}$$

which is (almost accidentally) correct. But since it's multiple choice, the incorrect subtraction $\infty - \infty = 0$ won't get noticed.

Another question from the same exam

"Show that

$$\int_{x=0}^{\infty} e^{-ax} \sin(x)\, dx = \frac{1}{1+a^2}$$

if $a > 0$."

This is a familiar sort of question. It could be asking for the use of a convergence test, and the standard two-step integration by parts. It is the "T" case from the LIATE rule, if you know that rule. (Logarithmic, Inverse trig, Algebraic, Trigonometric, Exponential: which to choose as u in $\int u dv = uv - \int v du$)

Maple's answer (if we don't ask the right way)

```
> integrate( exp(-a*x)*sin(x), x=0..infinity );
```

$$\lim_{x \to \infty} \left(-\frac{a\,\mathrm{e}^{-ax} \sin(x) + \mathrm{e}^{-ax} \cos(x) - 1}{a^2 + 1} \right)$$

That doesn't seem helpful. It doesn't *look* like an answer.

SymPy's answer

I skip some setup with SymPy, importing things and defining x as a symbol.

```
f = exp(-a*x)*sin(x)
integrate( f, (x, 0, oo) )
```

$$\begin{cases} \frac{1}{a^2+1} & \text{for } \left|\arg(a)\right| < \frac{\pi}{2} \\ \int_0^\infty e^{-ax} \sin(x)\, dx & \text{otherwise} \end{cases}$$

Hmm. Will first year students know what "arg" means? Luckily, we can guess, and this reminds us that we didn't tell either Maple or SymPy that $a > 0$. [And the second answer, that's weird, isn't it?]

Assumptions in Maple or SymPy

To tell Maple that $a > 0$ we can issue the command

```
>int(exp(-a*x)*sin(x),x=0..infinity) assuming a>0;
```

in which case it returns $1/(1+a^2)$ as expected.

In SymPy, we can add the attribute `positive=True` to the declaration of a as a symbol:

```
a = symbols('a', positive=True )
```

Again if we do this, the integration command then returns $1/(1+a^2)$ as expected.

Reflections

- A student quotation: “What *good* does [the technology] do? I mean, I *liked* plug-and-chug. Now I have to think about what the answer *means*!” (**activity** is pleasant)
- The problem of case dissection [link] is *hard*; there can be a combinatorial explosion of cases. The program needs the user to tell it about the assumptions on the variables.
- Using technology requires training. (Neither of those syntaxes was obvious to me before I learned them)
- We need to re-think our exam questions in the light of student needs, which have changed with the changing environment.

The case of calculus is discussed in more detail in my 2004 paper Computer-Mediated Thinking [link].

Is technology disheartening?

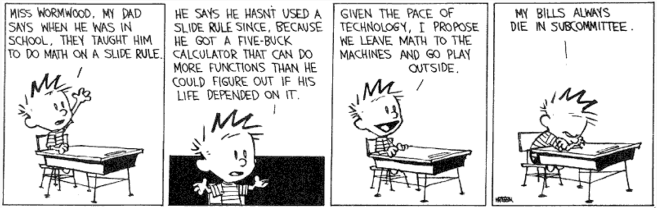

Figure 1: Hard to compete with a machine

Rethinking that integration question

The question was:

"Show that

$$\int_{x=0}^{\infty} e^{-ax} \sin(x)\, dx = \frac{1}{1+a^2}$$

if $a > 0$."

What could we be wanting to test, here?

1. The two-step integration by parts process (but, why?)
2. How to compare integrals to test for convergence (but, why?)
3. Laplace Transform (but, why?)
4. Can they write their answer using full, grammatical sentences (a mini-essay)? (I like this one)

Better questions?

a) to *formulate* this integral from an applied problem (Can test their writing skills this way)
b) how to use Maple or SymPy to evaluate this integral

(But then we have to test them on their CAS knowledge somehow)

Let's look at Linear Algebra

Again let's trawl the web. We find the following question from a 2022 Math 1600 exam at Western (no calculators or cell phones allowed):

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} . \tag{1}$$

This question is obsolete in at least two ways. First, "nearly anything that can be done with the matrix inverse can be done without it." Second, who inverts 3 by 3 matrices by hand nowadays?

Maybe a better question: Does this matrix factor $A = LU$ without pivoting? **[can't tell without doing it]** Matrix factorings are *far* more useful than matrix inverses.

> *"Linear algebra is the first course where the student encounters algebra, analysis, and geometry all at once together."*

—William (Velvel) Kahan,
to RMC at the 4th SIAM Linear Algebra Conference in Minneapolis 1991

See Kahan's paper Mathematics Written in Sand [Link] for an early and prescient view of the use of computational environments as "computational laboratories."

He envisaged the student as being an active explorer.

Challenges from IEEE floats

- “Admit, for instance, the existence of a minimum magnitude, and you will find that the minimum which you have introduced, small as it is, causes the greatest truths of mathematics to totter.” — Aristotle
- Floats are *not associative*: $a + (b + c) \neq (a + b) + c$ necessarily. For instance $-M + (M + 1) = 0$ while $(-M + M) + 1 = 1$ if $M = 3.14 \cdot 10^{17}$.
- This (and other features) break students’ models of how the world works.
- We have to enable students to deal with floats.

Some topics only make sense in a floating-point context

My “most popular” YouTube video is on Modified Gram-Schmidt [Link]

This topic has hardly any importance, except that it (unlike Classical Gram Schmidt) is fairly stable numerically.

Buchberger's White Box/Black Box

In his 1990 paper Buchberger outlines the *White Box/Black Box* principle for teaching mathematics in a mechanized environment.

- When learning a particular concept the 1st time, say determinant, the student is not allowed to use the `Determinant` routine
- When using the concept in learning more advanced things (eg Cramer's Rule) they *are* allowed to use it
- Why? People need a certain amount of human action to internalize a concept
- At that point we say the concept has become an **answer** and not a **question**

Sometimes the rule can be broken profitably; one can use Black Box for a while, then open up the box and see what's inside it.

Again, Buchberger is thinking of the student as being *active*.

We have already changed

Because of Wolfram Alpha [Link] homework assignments already have to be different. Then there's Chegg and MathOverflow and Maple Primes and many more. This puts more weight on exams. I am not even going to talk about ChatGPT!

Examining *with* technology is more stressful for the student. We have had lab equipment failures, software failures, and lots more. It's harder to invigilate, too.

But going the no-technology route requires banning cell phones (illegal in some countries). And limits the kind of question you can ask.

A possibly better exam question

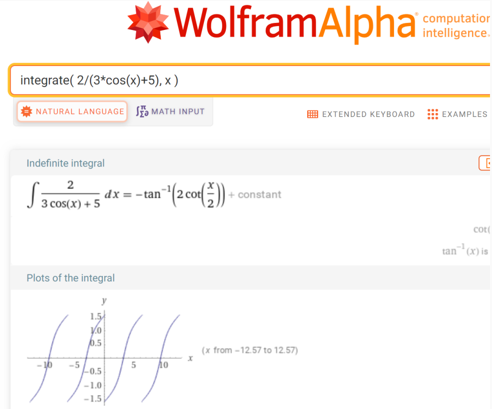

Figure 2: Question: Wolfram Alpha gives the output above. Is it correct?

Mandelbrot polynomials and Automatic Differentiation

```
function [p,dp]=mandelpoly(z,k)
% MANDELPOLY evaluates the k^th Mandelbrot polynomial
% and its derivative at one or more points.
% The k^th polynomial has degree 2^(k-1)-1

% Author PWL 2014.4.28 Modified RMC 2020.2.27
    dp = zeros(size(z));
    p  = zeros(size(z));
    for i=1:k-1
        dp = p.^2+2*z.*p.*dp;
        p  = z.*p.^2+1;
    end
```

The sixth iterate, $p_6(x)$, for real x

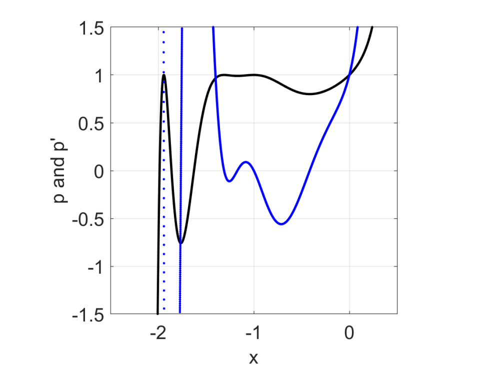

Figure 3: Graph of $p_6(x)$ and its derivative (blue) $p_6'(x)$ where $p_0(x) = 0$ and $p_{n+1}(x) = xp_n^2(x) + 1$.

Automatic Differentiation

Automatic Differentiation (also known as Program Differentiation) is important in industry and science. It's also a good topic to teach with, because it is quite rich (connection to dynamical systems, to complexity, to matrices, . . .).

Mathematical Notions Strengthened by Programming

Several mathematical notions are strengthened by these exercises.

- One can use *mathematical induction* to prove correctness of the automatic differentiation of the Mandelbrot polynomials
- The analysis of IEEE floats uses the IEEE guarantees ($\mathrm{fl}(x \mathrm{\ op\ } y) = (x \mathrm{\ op\ } y)(1+\delta)$ for some $|\delta| \leq \mathbf{u}$ where $\mathbf{u}$ is the *unit roundoff*, 2^{-53} for double precision)
- Practice with functions is always useful
- Simply working with visualizations improves people's feel for geometry.

Unintentional Power

Tampering with the Calculus course and the first Linear Algebra course will have significant downstream effects. Who will benefit from the changes? Will anyone be harmed? And do we really *have* to change?

Old-fashioned lecturing Calculus and Linear Algebra teaches students:

- To sit still for 50 minutes
- To take notes (gives practice writing or typing)
- Many other things (maybe gives them practice in arithmetic, logic, or algebra)

But the Big Ideas can be taught better with technology

- Functions
- Continuity (Carathéodory uses this to define differentiability)
- Limits and Convergence
- Reductionism

$$\int_a^b f(x)\,dx = \lim_{?} \sum_{k=1}^{N} f(x_k)\Delta x_k$$

where the limit is over various allowable partitions.

"Break the varying problem up into parts so small that on each of them the simple constant-coefficient law holds."

Differential equations (might not get time for this!)

I will “pick on” a book, this time, one that I reviewed thirty years ago: Abell & Braselton’s Differential Equations with Maple. [Link to my review]. I was, shall we say, blunter with my words when I was younger: “The authors failed to take advantage of their opportunities.”

Continued

Some of the opportunities they missed:

- Talking about modelling with differential equations
- Visualization with explanation and commentary
- Activity and Interactivity
- Verification of their results (there were too many mistakes)
- Exploration of nonuniqueness of solutions, of asymptotic behaviour, of geometric features and properties
- the opportunity to **throw away a topic**, namely power series solutions

How I think it *should* be done (a whole other one hour talk):

Variations on a Theme of Euler [Link]

Following up on that symbolically [Link]

All they did really was take a standard approach and graft Maple onto it.

Throw away a topic??

Poll: When was the last time you solved a differential equation in series, by hand (maybe using the method of Frobenius)?

What pedagogical value does teaching students this method have?

Possible answers: Analytic continuation (to explain numerical methods); Existence and uniqueness theorems. Others? Okay, it's something they can *do*, and be active with. But why?

On the other hand, if *that* topic is not removed, what else will be removed, in order to make room for numerical methods and training in their use? Or symbolic methods, come to that?

In Summary

- The environment in which our students use mathematics has changed
- The mathematics we teach must change with that
- Not only the methods we use to teach mathematics (videos seem extraordinarily popular) but also the **topics must change**
- Students need active training in the responsible use of technology
- That means we have to keep up with the technology
- The reactionaries are winning in some places now, but they should not be allowed to win overall

Thank you

Thank you for listening!

MINISTERIO
DE CIENCIA
E INNOVACIÓN

This work was partially supported by NSERC grant RGPIN-2020-06438, and partially supported by the grant PID2020-113192GB-I00 (Mathematical Visualization: Foundations, Algorithms and Applications) from the Spanish MICINN. I also thank CUNEF Universidad for financial support.

References

[1] Harold Abelson. Computation in the undergraduate curriculum. *International Journal of Mathematical Education in Science and Technology*, 7(2):127–131, 1976.

[2] Luis von Ahn. Duolingo: learn a language for free while helping to translate the web. In *Proceedings of the 2013 international conference on Intelligent user interfaces*, pages 1–2, 2013.

[3] Robert M Aiken and Richard G Epstein. Ethical guidelines for AI in education: Starting a conversation. *International Journal of Artificial Intelligence in Education*, 11:163–176, 2000.

[4] Aristotle. *On the Heavens*. 350BCE.

[5] Robert L Armacost and Julia Pet-Armacost. Using mastery-based grading to facilitate learning. In *33rd Annual Frontiers in Education, 2003. FIE 2003.*, volume 1, pages T3A–20. IEEE, 2003.

[6] Jack Betteridge, James H Davenport, Melina Freitag, Willem Heijltjes, Stef Kynaston, Gregory Sankaran, and Gunnar Traustason. Teaching of computing to mathematics students: Programming and discrete mathematics. In *Proceedings of the*

3rd Conference on Computing Education Practice, pages 1–4, 2019.

[7] Jack Betteridge, Eunice Y. S. Chan, Robert M. Corless, James H. Davenport, and James Grant. Teaching programming for mathematical scientists. In *Mathematics Education in the Age of Artificial Intelligence*, pages 251–276. Springer International Publishing, 2022. URL *https://doi.org/10.1007/978-3-030-86909-0_12*.

[8] P Bond. The era of mathematics–review findings on knowledge exchange in the mathematical sciences. engineering and physical sciences research council and the knowledge transfer network. *https://epsrc.ukri.org/newsevents/news/mathsciencereview/*, 2018.

[9] J. Borwein and K. Devlin. The computer as crucible: An introduction to experimental mathematics. *The Australian Mathematical Society*, page 208, 2009.

[10] J. M. Borwein and P. B. Borwein. Strange series and high precision fraud. *The American mathematical monthly*, 99(7):622–640, 1992.

[11] Paul L. Boynton. What constitutes good teaching? *Peabody Journal of Education*, 28(2):67–73, 1950. ISSN 0161956X. URL `http://www.jstor.org/stable/1489824`.

[12] R.J. Bradford, J.H. Davenport, and C.J. Sangwin. A Comparison of Equality in Computer Algebra and Correctness in Mathematical Pedagogy. In J. Carette *et al.*, editor, *Proceedings Intelligent Computer Mathematics*, pages 75–89, 2009.

[13] Meredith Broussard. *Artificial unintelligence: How computers misunderstand the world*. MIT Press, 2018.

[14] Bruno Buchberger. Should students learn integration rules? *ACM Sigsam Bulletin*, 24(1):10–17, 1990.

[15] Neil J. Calkin, Eunice Y.S. Chan, and Robert M. Corless. *Computational Discovery on Jupyter*. SIAM, 2023 (in progress). URL `https://computational-discovery-on-jupyter.github.io/Computational-Discovery-on-Jupyter/`.

[16] A.C. Camargos Couto, M. Moreno Maza, D. Linder, D.J. Jeffrey, and Corless R.M. Comprehensive LU Factors of Polynomial Matrices. *MACIS 2019*, pages 80–88, 2020.

[17] David Carlson, Charles R. Johnson, David Lay, and A. Duane Porter. Gems of exposition in elementary linear algebra. *The College Mathematics Journal*, 23(4):299–303, September 1992. doi: 10.1080/07468342.1992.11973473.

[18] David Carlson, Charles R Johnson, David C Lay, and A Duane Porter. The linear algebra curriculum study group recommendations for the first course in linear algebra. *The College Mathematics Journal*, 24(1):41–46, 1993.

[19] Eunice Y. S. Chan and Robert M Corless. A random walk through experimental mathematics. In *Jonathan M. Borwein Commemorative Conference*, pages 203–226. Springer, 2017.

[20] R. Chapman and F. Schanda. Are We There Yet? 20 Years of Industrial Theorem Proving with SPARK. In *Proceedings of Interactive Theorem Proving*, pages 17–26, 2014.

[21] R. M. Corless and J. E. Jankowski. Variations on a theme of Euler. *SIAM Review*, 58(4):775–792, 2016.

[22] R. M. Corless and D. J. Jeffrey. Scientific computing: One part of

the revolution. *Journal of Symbolic Computation*, 23(5):485–495, 1997.

[23] R. M. Corless, C. Essex, and P. J. Sullivan. *First year engineering mathematics using supercalculators*. SciTex, The University of Western Ontario, 2 edition, 1995.

[24] Robert M Corless. Six, lies, and calculators. *The American mathematical monthly*, 100(4):344–350, 1993.

[25] Robert M Corless. Computer-mediated thinking. *Proceedings of Technology in Mathematics Education*, 2004. `https://github.com/rcorless/rcorless.github.io/blob/main/CMTpaper.pdf`.

[26] Robert M Corless and David J Jeffrey. The Turing factorization of a rectangular matrix. *ACM SIGSAM Bulletin*, 31(3):20–30, 1997.

[27] Robert M Corless, David J Jeffrey, and David R Stoutemyer. Integrals of functions containing parameters. *The Mathematical Gazette*, 104(561):412–426, 2020.

[28] Robert M. Corless, David J. Jeffrey, and Azar Shakoori. Teaching linear algebra in a mechanized mathematical environment, 2023. Accepted to CICM 2023, Cambridge.

[29] James H Davenport, David Wilson, Ivan Graham, Gregory Sankaran, Alastair Spence, Jack Blake, and Stef Kynaston. Interdisciplinary teaching of computing to mathematics students: Programming and discrete mathematics. *MSOR Connections*, 14(1):1–8, 2014.

[30] James H. Davenport, Alan Hayes, Rachid Hourizi, and Tom Crick. Innovative pedagogical practices in the craft of computing. In *2016 International Conference on Learning and Teaching in Computing and Engineering (LaTICE)*, pages 115–119. IEEE, 2016.

[31] J.H. Davenport. Methodologies of Symbolic Computation. In *Proceedings AISC 2018*, pages 19–33, 2018.

[32] F. Fischer, K. Böttinger, H. Xiao, C. Stransky, Y. Acar, M. Backes, and S. Fahl. Stack Overflow Considered Harmful? The Impact of Copy&Paste on Android Application Security. *38th IEEE Symposium on Security and Privacy (SP)*, pages 121–136, 2017.

[33] Michael Frame and Benoit B. Mandelbrot. *Fractals, graphics, and mathematics education*. Number 58. Cambridge University Press, 2002.

[34] Scott Freeman, Sarah L Eddy, Miles McDonough, Michelle K Smith, Nnadozie Okoroafor, Hannah Jordt, and Mary Pat Wenderoth. Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, 111(23):8410–8415, 2014.

[35] Laureano Gonzalez-Vega. Using linear algebra to introduce computer algebra, numerical analysis, data structures and algorithms (and to teach linear algebra, too). *International Journal of Computer Algebra in Mathematics Education*, 6(3):209, 1999. ISSN 1362-7368. URL `https://www.learntechlib.org/p/92417`.

[36] R. W. Hamming. *Methods of mathematics applied to calculus, probability, and statistics*. Courier Corporation, 2012.

[37] J. Handelsman, D. Ebert-May, R. Beichner, P. Bruns, A. Chang,

R. DeHaan, J. Gentile, S. Lauffer, J. Stewart, S. M. Tilghman, et al. Scientific teaching. *Science*, 304(5670):521–522, 2004.

[38] Stephen Hegedus, Colette Laborde, Corey Brady, Sara Dalton, Hans-Stefan Siller, Michal Tabach, Jana Trgalova, and Luis Moreno-Armella. *Uses of technology in upper secondary mathematics education.* Springer Nature, 2017.

[39] Nicholas J. Higham. *Accuracy and Stability of Numerical Algorithms.* SIAM, Philadelphia, 2nd edition, 2002.

[40] David J Jeffrey and Robert M Corless. Linear algebra in Maple®. In Leslie Hogben, editor, *Handbook of Linear Algebra*, pages 89–1. Chapman and Hall/CRC, 2nd edition, 2013.

[41] William M. Kahan. Handheld calculator evaluates integrals. *Hewlett-Packard Journal*, 31(8):23–32, 1980.

[42] William M. Kahan. Mathematics written in sand. In *Proc. Joint Statistical Mtg. of the American Statistical Association*, pages 12–26, 1983. http://people.eecs.berkeley.edu/~wkahan/MathSand.pdf.

[43] Zoltán Kovács, Tomás Recio, Philippe R Richard, and M Pilar Vélez. GeoGebra automated reasoning tools: A tutorial with examples. In *Proceedings of the 13th International Conference on Technology in Mathematics Teaching*, pages 400–404, 2017.

[44] Zoltán Kovács, Tomás Recio, and M. Pilar Vélez. Automated reasoning tools with GeoGebra: What are they? what are they good for? In *Mathematics Education in the Age of Artificial Intelligence*, pages 23–44. Springer International Publishing, 2022. doi: 10.1007/978-3-030-86909-0_2. URL https://doi.org/10.1007/978-3-030-86909-0_2.

[45] David C Lay, Steven R Lay, and Judi McDonald. *Linear algebra and its applications*. Pearson Education, 2016.

[46] Ao Li and Robert M Corless. Revisiting Gilbert Strang's "A chaotic search for *i*". *ACM Communications in Computer Algebra*, 53(1): 1–22, 2019.

[47] Rose Luckin, Wayne Holmes, Mark Griffiths, and Laurie B Forcier. *Intelligence unleashed: An argument for AI in education*. Pearson Education, 2016. ISBN 9780992424886.

[48] Benoit B. Mandelbrot and Michael Frame. Some reasons for the effectiveness of fractals in mathematics education. *Fractals, Graphics, & Mathematics Education*, pages 3–9, 2002.

[49] Cleve Moler. Roots—of polynomials, that is, 1991. URL *https://www.mathworks.com/company/newsletters/articles/roots-of-polynomials-that-is.html*.

[50] John Monaghan, Luc Trouche, and Jonathan M Borwein. *Tools and mathematics*. Springer, 2016.

[51] S. Papert. *Mindstorms*. Basic Books, 2 edition, 1993.

[52] John Paxton. Live programming as a lecture technique. *Journal of Computing Sciences in Colleges*, 18(2):51–56, 2002.

[53] Christopher Rackauckas and Qing Nie. Differentialequations. jl–a performant and feature-rich ecosystem for solving differential equations in Julia. *Journal of open research software*, 5(1), 2017.

[54] Peter A. Rosati, Robert M. Corless, G. Christopher Essex, and Paul J. Sullivan. An evaluation of the HP28S calculator in calculus. *Australian J. Engineering Education*, 3(1):79–88, 1992.

[55] Marc J Rubin. The effectiveness of live-coding to teach introductory programming. In *Proceeding of the 44th ACM technical symposium on Computer science education*, pages 651–656, 2013.

[56] Elaine Seymour and Nancy M Hewitt. *Talking about leaving*. Westview Press, Boulder, CO, 1997.

[57] J.R. Slagle. A Heuristic Program that Solves Symbolic Integration Problems in Freshman Calculus. *Journal of the ACM*, 10:507–520, 1963.

[58] Richard C Smith and Edwin F Taylor. Teaching physics on line. *American Journal of Physics*, 63(12):1090–1096, 1995.

[59] G. Strang. Too much calculus. `http://www-math.mit.edu/~gs/papers/essay.pdf`, 2001.

[60] Gilbert Strang. *Introduction to Applied Mathematics*. Wellesley-Cambridge Press, Wellesley, MA, 1986.

[61] Diane Hobenshield Tepylo and Lisa Floyd. Learning math through coding. 2016. *https://researchideas.ca/mc/learning-math-through-coding/*.

[62] Lloyd N Trefethen and David Bau. *Numerical linear algebra*, volume 181. Siam, 2022.

[63] Charles F. Van Loan and K.-Y. Daisy Fan. *Insight Through Computing - A MATLAB Introduction to Computational Science and Engineering*. SIAM, 2010. ISBN 978-0-89871-691-7.

[64] Greg Wilson. Software carpentry: getting scientists to write better code by making them more productive. *Computing in Science & Engineering*, 8(6):66–69, 2006.

[65] Greg Wilson, Dhavide A Aruliah, C Titus Brown, Neil P Chue Hong, Matt Davis, Richard T Guy, Steven HD Haddock, Kathryn D Huff, Ian M Mitchell, Mark D Plumbley, et al. Best practices for scientific computing. *PLoS Biol*, 12(1):e1001745, 2014.