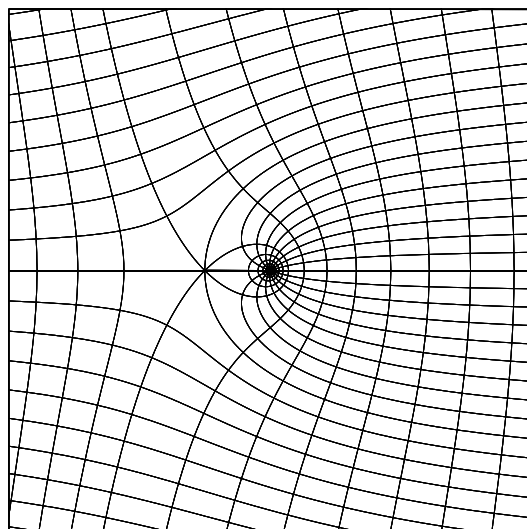# The Applied Mathematics Coffee Mug

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The design on the Department of Applied Mathematics "official coffee mug" is, on one side, the crest of the University of Western Ontario, and on the other side, the following picture.

This picture is the image of circles and radial lines under the conformal map $z \mapsto W(z)$, where $W(z)$ is the Lambert W function, which is of surprisingly widespread current interest [1]. This function is the (multivalued) inverse of $w \mapsto w \exp w$, so it satisfies

$$W(z)e^{W(z)} = z \,. \tag{1}$$

Because this is such a simple transcendental equation, it occurs in a very wide variety of applications, from delay differential equations to the analysis of algorithms.

Equivalently, then, this picture is the *inverse* image of circles and radial lines under the map $w \mapsto w \exp w$.

Parametric Description

The image was constructed by means of a parametric solution. We put

$$z = re^{i\theta} \,,$$

and if $w = u + iv$, then $z = w \exp(w)$ implies

$$re^{i\theta} = (u + iv)e^{u+iv} \tag{2}$$

or

$$u + iv = re^{-u}e^{i(\theta - v)}\,.\tag{3}$$

Separating real and imaginary parts we have

$$
\begin{aligned}
u &= re^{-u}\cos(\theta - v) \tag{4}\\
v &= re^{-u}\sin(\theta - v)\,. \tag{5}
\end{aligned}
$$

Dividing equation (4) by equation (5) we get, if $v \neq 0$,

$$u = v\cot(\theta - v)\,.\tag{6}$$

The case $v = 0$, which turns out to be an image of $-1/e \leq z < \infty$, is left as an exercise. Squaring and adding equations (4) and (5), we also get

$$u^2 + v^2 = r^2 e^{-2u}\,,$$

or

$$v = \pm\sqrt{(r\exp(-u) - u)(r\exp(-u) + u)}\,.\tag{7}$$

To plot an image of a radial line, with constant θ, we choose a range of values of v and plot equation (6). To plot an image of a circular arc, with constant r, we choose a range of u and plot equation (7). This equation is difficult to plot accurately when $r^2\exp(-2u) - u^2$ is small, unless a lot of points are used. A superior parametric description of the curve in this region is

$$
\begin{aligned}
u &= W(rs) \tag{8}\\
v &= \pm\frac{\sqrt{1 - s^2}}{s}W(rs)\,, \qquad \text{for } -1 \leq s \leq 1\,, \tag{9}
\end{aligned}
$$

the derivation of which we leave as an exercise. [Hint: put $u = tv$ for some parameter t, and then later change parameters to $s = t/\sqrt{1 + t^2}$.] Note that both real branches of W must be used to get all the lines. Nothing is perfect, however, and this parameterization requires a lot of points near $rs = -1/e$.

The design on the mug was done by choosing radii of circles, and radial lines, that were spaced widely enough that the resulting picture was aesthetically pleasing. The most important lines, namely the images of the circle with radius $1/e$ and the radial line $\theta = \pm\pi$, were naturally included. This gives the crossing at $w = -1$. At every other point except $w = 0$, we have orthogonal crossings; because the branch point for $W(z)$ is a second-order point, at $z = -1/e$ or $w = -1$, the angles of the crossings at $w = -1$ are $\pm\pi/4$ or $\pm45°$.

Acknowledgements

References

[1] Robert M. Corless, Gaston H. Gonnet, David E. G. Hare, David J. Jeffrey, and Donald E. Knuth, "On the Lambert W function", Advances in Computational Mathematics, volume 5, 1996, pp. 329–359.